IB Linear Algebra – Example Sheet 3

Michaelmas 2024 `hb470@cam.ac.uk`

1. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$
The second and third matrices commute; find a basis with respect to which they are both diagonal.

2. By considering the rank or minimal polynomial of a suitable matrix, find the eigenvalues of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. Hence write down the determinant of A.

3. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimum polynomial of A, over the complex numbers, has real coefficients.

4. Let α be an endomorphism of the real vector space V satisfying the equation $\alpha^3 = \alpha$. For each $\lambda \in \{-1, 0, 1\}$, find a linear map $\pi_\lambda : V \to V$ (depending on α) such that:
   (i) $\mathrm{Im}(\pi_\lambda) \leq V_\lambda$ (the λ-eigenspace of α);
   (ii) $\mathrm{id}_V = \pi_{-1} + \pi_0 + \pi_1$;
   (iii) For $\lambda \neq \mu$, $\pi_\lambda \circ \pi_\mu = 0$.

   Use the π_λ to prove that $V = V_0 \oplus V_1 \oplus V_{-1}$.

5. Let α be an endomorphism of a complex vector space V. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. Are the eigenspaces $\mathrm{Ker}(\alpha - \lambda \mathrm{id}_V)$ and $\mathrm{Ker}(\alpha^2 - \lambda^2 \mathrm{id}_V)$ necessarily the same?

6. (Cayley–Hamilton theorem for a general field) Let $\mathbb{F}$ be any field, $A \in M_{n\times n}(\mathbb{F})$, and $B = tI - A$. By considering the equation $B \cdot \mathrm{adj}\,(B) = \det(B)I$, or otherwise, prove that $\chi_A(A) = 0$, where $\chi_A(t)$ is the characteristic polynomial of A.

7. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in Jordan normal form. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^{1000}$.

8. Without appealing directly to the uniqueness of Jordan Normal Form show that none of the following matrices are similar:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Is the matrix
$$\begin{pmatrix} -2 & -2 & -1 \\ 3 & 3 & 1 \\ 3 & 2 & 2 \end{pmatrix}$$
similar to any of them? If so, which? Find a basis such that it is in Jordan Normal Form.

9. (a) Recall that the Jordan normal form of a 3×3 complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices.
(b) Let A be a 5×5 complex matrix with $A^4 = A^2 \neq A$. What are the possible minimal polynomials of A? If A is not diagonalisable, what are the possible characteristic polynomials and JNFs of A? *(The list is quite long!)*

10. Let V be a vector space of dimension n and α an endomorphism of V with $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$. Without appealing to Jordan Normal Form, show that there is a vector y such that $(y, \alpha(y), \alpha^2(y), \ldots, \alpha^{n-1}(y))$ is a basis for V. What is the matrix representation of α with respect to this basis? And the matrix representation of α^k, for an arbitrary positive integer k?

    Show that if β is an endomorphism of V which commutes with α, then $\beta = p(\alpha)$ for some polynomial p. *[Hint: consider $\beta(y)$.]* What is the form of the matrix for β with respect to the above basis?

11. (a) Let A be an invertible square matrix. Describe the eigenvalues and the characteristic and minimal polynomials of A^{-1} in terms of those of A.

    (b) Prove that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan Normal Form a Jordan block $J_m(\lambda^{-1})$. Use this to find the Jordan Normal Form of A^{-1}, for an invertible square matrix A.

    (c) Prove that any square complex matrix is similar to its transpose.

12. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

13. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_n & a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

    Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.